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A New Approach to the Family Structure.

Otto C. W. Kong

Institute of Field Physics, Department of Physics and Astronomy,

University of North Carolina, Chapel Hill, NC 27599-3255

Abstract

In this letter, we introduce a new approach to formulate the family structure of the standard model. Trying to mimic the highly contrained representation structure of the standard model while extending the symmetry, we propose a $SU(4) \otimes SU(3) \otimes SU(2) \otimes U(1)$ symmetry with a SM-like chiral spectra basically "derived" from the gauge anomaly constraints. Embedding the SM leads to $SU(4)_A \otimes SU(3)_C \otimes SU(2)_L \otimes U(1)_X$ models, which upon the $SU(4)_A \otimes U(1)_X \longrightarrow U(1)_Y$ symmetry breaking, gives the three families naturally as a result. A specific model obtained from the approach is illustrated. The model, or others from our approach, holds promise of a very interesting phenomenology. We sketch some of the results here. An interesting possiblity of supersymmetrizing the model with the EW-Higgses already in the spectrum is noted. A comparison with other approaches is also discussed.

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In this letter, we introduce a new approach to the much pursued, however still mysterious, family problem. Trying to mimic the highly contrained group representation structure of the standard model while extending the symmetry, we describe a specific $SU(4)_A \otimes SU(3)_C \otimes SU(2)_L \otimes U(1)_X$ model which, upon symmetry breaking, gives the three-family standard model together with some extra vector-like states. The fully chiral fermion spectrum of the model, is largely "derivable" from the gauge anomaly constraints, like the SM. Details of the representation content of the SM-like chiral spectrum are given in table 1. To set the stage and facilitate comparison, we first note the major available approaches, before we describe our new alternative.

The matter content of the phenomenologically very successful standard model(SM) consists of three sets (families) of 15 chiral fermion states of identical quantum numbers. In addition, a scalar doublet Higgs is needed to break the electroweak(EW) symmetry and give the fermions masses. The existence of the scalar leads to the hierarchy problem, which is widely believed to be addressed by supersymmetry(SUSY). The representation structure of a single family of chiral fermions is very strongly constrained by the requirement of cancellation of all gauge anomalies, making it easily "derivable" once some simple assumptions are taken [1]. However, the existence of *three* families, with the great hierarchy of masses among the fermions after EW-symmetry breaking, remains a mystery.

There have been many attempts on explaining the family structure since the late 70's. The simplest approach is to introduce an extra family or horizontal symmetry commuting with the standard model group or its (vertical) unification group [2]. It is desirable to have a gauged horizontal symmetry since anomaly constraints reduce much of the arbitrariness in the model building exercises. Moreover, analysis of gravitational effect [3] casts a strong doubt on the consistence of global horizontal symmetry models, at least for those that have a high symmetry breaking scale. Recently, there are a lot of activities on this kind of model building [4], aiming at obtaining phenomenologically viable textures for the quark, and the less constrained squark, mass matrices [5].

There are more non-trivial ways of putting the family structure into a gauge symmetry.

For instance the very nice embedding of the one family SM into a $SU(5)$ unification [6] had motivated further unification models that lead to three chiral families [7–9]. An illustrative example [8] has the following structure: the symmetry breaking is assumed to be $SU(9) \longrightarrow SU(5) \longrightarrow SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$; an anomaly free fermion content, $9[\mathbf{9},\mathbf{8}] + [\mathbf{9},\mathbf{3}]$, is taken which then results in three chiral $SU(5)$ families. There is no apparent way to build the fermion mass hierarchy among the families. Moreover, the family structure is built in above the remote unification scale. Unlike the horizontal symmetry approach, this one is not much pursued for quite some time.

Yet another interesting alternative is proposed by Frampton [10] relatively recently, called the 331-model. There $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ of the SM is embedded into $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ with a nontrivial hypercharge embedding. The three families are put in separately; however, the third family has a representation structure different from that of the first two and gauge anomalies cancelled among the three families. Extra structure is needed, to produce the symmetry breaking and give the exotic states reasonably large mass.

Now, let us take a look at the elegance of the SM representation structure for a single family. For example:

- We can start by introducing the simplest multiplet that transforms nontrivially under each of the component group factors, namely a $(\mathbf{3},\mathbf{2},\mathbf{1})$. (Here fixing the hypercharge as 1 just corresponds to a arbitrary choice of normalization.)

- To cancel the $SU(3)$ anomaly, two $\bar{\mathbf{3}}$'s are needed. Keeping with the chiral structure, we have to use a $(\bar{\mathbf{3}},\mathbf{1},\mathbf{x})$ and a $(\bar{\mathbf{3}},\mathbf{1},\mathbf{y})$.

- Next, cancellation of the global-$SU(2)$ anomaly dictates the inclusion of an extra doublet, $(\mathbf{1},\mathbf{2},\mathbf{a})$.

- We still have to cancel all the $U(1)$-anomalies. We have $a = -3$ from the $[SU(2)]^2U(1)$ anomaly constraint. x and y then has to satisfy the three remaining contraints and no solution can be found. If we then allow one singlet state, $(\mathbf{1},\mathbf{1},\mathbf{k})$, we have three equations for three unknowns giving a unique solution, the SM hypercharge assignment $(x, y = 2, -4; k = 6)$. Notice that the solution *a priori* may not give a set of rational numbers. The triviality of solution here is a bit deceiving.

We want to try to mimic the above feature in a extended symmetry that can incorporate the three families naturally. Enlarging one of the component group factors apparently does not work. We consider adding one more factor. Then $SU(4) \otimes SU(3) \otimes SU(2) \otimes U(1)$ suggests itself as the most natural candidate.

- We start with a $(\mathbf{4}, \mathbf{3}, \mathbf{2}, \mathbf{1})$. Develop along the same lines as above, we have the list:

$$(\mathbf{4}, \mathbf{3}, \mathbf{2}, \mathbf{1}), (\bar{\mathbf{4}}, \bar{\mathbf{3}}, \mathbf{1}, \mathbf{x}), (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{y}), (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{z}),$$
$$(\mathbf{1}, \bar{\mathbf{3}}, \mathbf{2}, \mathbf{a}), (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{1}, \mathbf{b}), (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{1}, \mathbf{c}),$$
$$(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{k}), (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{s}).$$

  Notice that all gauge anomalies not involving the $U(1)$ group cancel; in particular, there is an equal number of $\mathbf{4}$'s and $\bar{\mathbf{4}}$'s, or $\mathbf{3}$'s and $\bar{\mathbf{3}}$'s, and a even number of $\mathbf{2}$'s.

- If we asume we have a $SU(4)_A \otimes SU(3)_C \otimes SU(2)_L \otimes U(1)_X$, the list has basically what it takes to give the three chiral families after $SU(4)_A \otimes U(1)_X$ breaks into $U(1)_Y$. Replacing the $SU(4)$ by other $SU(N)$ groups may also be considered. $N < 4$, cannot accomodate three families, nor can $N > 6$. $N = 5$ and $N = 6$ present viable alternatives when given a bit "un-natural" representation content (see table 2). One can also consider a nontrivial embedding of $SU(3)_C$ or $SU(2)_L$. That would not work either, at least not for $N = 4, 5$ and 6 [11].

- Finding a set of X-charge assignments that can cancel all the related anomalies is highly nontrivial. While the author does have a successful solution, the approach may not be very fruitful. We have another issue on the agenda: building the correct hypercharge embedding that enables us to identify the SM families.

- There are three independent $U(1)$ factors in $SU(4)$. To embed $U(1)_Y$, we can identify it as a linear combination of the three and the extra $U(1)_X$. Remember that we want a three-family SM. For instance, there are four quark doublets contained in $(\mathbf{4}, \mathbf{3}, \mathbf{2}, \mathbf{1})$. We want the correct hypercharge for three of them; and the fourth better form a vector-like pair with $(\mathbf{1}, \bar{\mathbf{3}}, \mathbf{2}, \mathbf{a})$. Assuming SUSY, the chiral multiplets contain also scalar states. We can then assume a hypercharge invariant VEV for a scalar state in $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{z})$, that breaks $SU(4)_A \otimes U(1)_X$ into for example $SU(3)_H \otimes U(1)_Y$ with the $SU(3)_H$ as a horizontal symmetry for the SM. This fixes the quark doublets. Putting in all the requirements from the SM $U(1)_Y$ assignments, there is a unique solution for the set of X-charges and a specific $U(1)_Y$ definition as a linear combination of $U(1)_X$ and the other $U(1)$ from $SU(4)_A$, up to a two fold ambiguity in embedding the quark singlets.

- Finally, we have to check all the $U(1)_X$-anomalies. It sounds like we need a miracle to have all the gauge anomalies just cancelled. It does *not* work! However, a small modification of the rules of the game does give us what we want: a consistent gauge model of three SM families together with the extra states all forming vector-like pairs after the symmetry breaking! We just need to add a $(\mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{p})$, which is free of $SU(4)$-anomaly, and some more singlets. The detailed structure of a specific model is listed in table 1.

We just sketched above our model construction approach. Apart from the two embedding alternatives, there is still some flexibility in modifying the leptonic part of the spectrum. These, together with a detailed discussion of our construction and some phenomenological features, we leave to another publication [12]. We give below an outline of some of the results presented there. The specific model is used to illustrate the basic features of models from our approach.

We have three SM quark doublets from $(\mathbf{4}, \mathbf{3}, \mathbf{2}, \mathbf{1})$, which also contains, together with $(\mathbf{1}, \bar{\mathbf{3}}, \mathbf{2}, \mathbf{-10})$, a fourth vector-like quark doublet(Q') with electric charges $(-1/3, -4/3)$.

The singlet quark states have interesting embeddings. The three $\bar{u}$'s and one $\bar{d}$ are in $(\bar{\mathbf{4}}, \bar{\mathbf{3}}, \mathbf{1}, \mathbf{5})$ while the other two $\bar{d}$'s are the $(\mathbf{1}, \bar{\mathbf{3}}, \mathbf{2}, -\mathbf{4})$'s. This difference in structure between the up- and down-sector might explain the expected difference in the textures of the two mass matrices [5] and hence the very existence of the CKM-matrix.

The scalar $\phi_0 = (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{9})$ with the natural $SU(4)_A \otimes U(1)_X$ breaking VEV

$$\langle \phi_0 \rangle = \begin{pmatrix} 0 & 0 & 0 & v \end{pmatrix} \tag{1}$$

can actually be used to define the remnant $SU(3)_H$ and $U(1)_Y$ symmetry. A possible scheme of quark mass generation then involves taking the Higgs doublets from a $\Phi = (\mathbf{15}, \mathbf{1}, \mathbf{2}, -\mathbf{6})$ together with extra SM singlet scalars from two $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{1}, -\mathbf{3})$'s, denoted by ϕ_a ($a = 1$ or 2), with natural VEVs

$$\langle \phi_1 \rangle = \begin{pmatrix} v_1 & 0 & 0 & 0 \end{pmatrix}, \qquad \langle \phi_2 \rangle = \begin{pmatrix} v_1' & v_2 & 0 & 0 \end{pmatrix}. \tag{2}$$

A mass term for Φ of the form [13]

$$C_{ab} \phi_{ai} \phi_b^{\dagger j} \Phi_j^k \Phi_k^{\dagger i}$$

then gives masses to all component doublets of the scalar except three, two of which contain zero electric charge states that can function as the EW-breaking Higgses. The two Higgses each couples directly to only one of SM quark; they give mass to only the top and the bottom. Hence, FCNC constraints [14] could be satisfied by assuming $v_i \geq 200TeV$. FCNC's from the extra, nine, neutral gauge bosons would also be under control then [15]. Smaller masses for the lighter quarks could be generated through other secondary mechanism or an enriched scalar sector, giving naturally hierarchial quark mass matrices. For example, in a non-SUSY setting, the given set of scalar VEVs can be combined to give suppressed effective mass terms to the second family, leaving the first family masses to be generated by radiative effects.

The third light doublet consists of Higgses of electric charges 1 and 2. The doubly charged Higgs is a novel prediction of the model. They couple the SM quarks to the Q'. The Higgs that is responsible for the top mass also gives mixing between the fourth down-type quark in

Q' and the bottom. This mixing fixes the R_b-anomaly [17] if $M_{Q'} \sim 715 GeV$. The scenario may be possible if value of $M_{Q'}$ is suppressed by small (effective) Yukawa coupling.

Upon QCD confinement, the new quarks will lead to interesting new mesonic and baryonic states. Notice that none of these has a fractional charge.

For the leptonic doublets, we have a $\bar{L}$ from $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{3})$ and four L's from the rest of $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{3})$ and the $(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{6})$. In a SUSY scenario, the extra pair of vector-like doublets might be identified as the Higgs(ino) doublets, instead of using Φ above, hence making our SM-like chiral spectrum self-contained; as the ϕ_0 scalar is already in the spectrum, as remarked above. The model then has lepton-number violation in a way similar to some recent analysis [16]. The details of the quark and lepton mass generations, and the scalar masses including the soft SUSY-breaking parts have however to be analyzed within the framework of our extended symmetry to see if the model could then be made consistent and realistic.

The SM-singlet sector is much richer. There is a right-handed neutrino state(N) in the $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{9})$, which can develop Majoarna mass invariant under the SM symmetry. The right-handed neutrino may give rise to neutrino masses through the see-saw mechanism. That may retrospectively be used to fix the symmetry breaking scale. There are also three extra vector-like singlet states of both electric charge $1(E/\bar{E})$ and $2(S/\bar{S})$. The former will be involved in mixing with the charged-leptons in the corresponding mass matrix.

While there is no obvious gauge group unification, string-type gauge coupling unification may not be ruled out. With SUSY assumed and the SM-like chiral spectrum taken without extra supermultiplets, the coefficients for the first order β-functions are given by $(b_4, b_3, b_2, b_1) = \frac{1}{16\pi^2}(-5, -1, 4, 233/24)$ where we have normalized the X-charge by $1/24$. We can see that the model just maintains the $SU(3)_C$ asymptotic freedom, a feature shared also by the $SU(4)_A$ component. We have to break $SU(4)_A$ before it confines to get the correct number of families. Assuming gauge coupling unification, this could be used to set a limit for the scale of the symmetry breaking. However, extra supermultiplets incorporating the symmetry breaking scalars, such as Φ used above, would have to be included in a realistic model. They would have very strong effect on the all the coefficient except b_3, easily removing the asymptotic freedom of $SU(4)_A$ for instance. The analysis is very important but cannot be done without a fully realistic model. this we will leave for further investigations.

Finally, we compare our approach with other formulations of a three-family SM. From the descriptions above, the major features are obvious. Our construction approach is unique. It has a SM-like chiral spectrum which is largely "derived", following similar features of the one family SM, and *yield three families as a natural result.* The model obtained from the approach, however, looks in some sense a hybrid of the three different major approaches used by previously authors. It has part of the extra symmetry similar to the horizontal symmetries. The major difference between the latter and our symmetry structure as an extension of the SM symmetry is that the hypercharge embedding is nontrivial in our case, leading to existence of extra charged gauge bosons. Like the other two approaches also mentioned at the beginning, we start with a fully chiral high energy fermion content, which then gives rise to the correct low energy chiral fermion content in a nontrivial way. However, it is not a grand unification type model. There is a possibility of embedding the $SU(4)_A \otimes SU(3)_C \otimes SU(2)_L \otimes U(1)_X$ group into a $SU(9)$ or some other simple groups of higher rank. No such unification scheme is apparent for incorporating the chiral spectrum. Because it is not tied up with unification, the symmetry breaking scale may not be remote, giving it plausible interesting phenomenology may be accessible by future experimental machines, like the 331-model [10]. The 331-model also shares a nontrivial hypercharge embedding. Our model contains 84 chiral fermionic states. This is to be compared with the 165 states from the $SU(9)$ model sketched above. Another similar minimal three family model with a $SU(7)$ symmetry has 133 states [9]. With only 63 states, the 331-model is still a bit more economic. However, the 84-state SM-like chiral spectrum, shares, to a great extent, the elegant structure of that of one SM family. Further investigations of the properties of the model, or others of the type, worth serious efforts.

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REFERENCES

[1] See P.H. Frampton and R.N. Mohapatra, Phys. Rev. **D50**, 3569 (1994), and references therein.

[2] S. Pakavasa and H. Sugawara, Phys. Lett. **73B**, 61 (1978); F. Wilczek and A. Zee, Phys. Rev. Lett. **42**, 421 (1979); Phys. Rev. **D25**, 553 (1982).

[3] S. Hawking, Phys. Lett. B **195**, 377 (1987); Phys. Rev. D **37**, 904 (1988); S.B. Giddings and A. Strominger, Nucl. Phys. **B306**, 890 (1988); *ibid.* **B307**, 854 (1988); S. Coleman, Nucl. Phys. **B307**, 867 (1988); *ibid.* **B310**, 643 (1988).

[4] See P.H. Frampton and O.C.W. Kong, Phys. Rev. **D53**, R2293 (1995); *IFP-725-UNC* (1996), hep-ph/9603372. to be published in Phys. Rev. Lett. **77**, no.5 (1996) ; and references therein.

[5] P. Ramond, R.G. Roberts and G. G. Ross, Nucl. Phys. **B406**, 19 (1993); R. Peccei and K. Wang, Phys. Rev. **D53**, 2712 (1995); P.H. Frampton and O.C.W. Kong, *IFP-724-UNC* (1996), hep-ph/9603371.

[6] S. Glashow and H. Georgi, Phys. Rev. Lett. **32**, 438 (1974); A. Zee, Phys. Lett. **99B**, 110 (1981); Y. Tosa and S. Okubo, Phys. Rev. **D23**, 2486 (1981); *ibid.* **D23**, 3058 (1981).

[7] H. Georgi, Nucl. Phys. **B156**, 126 (1979); P.H. Frampton and S. Nandi, Phys. Rev. Lett. **43**, 1460 (1979).

[8] P.H. Frampton, Phys. Lett. **89B**, 352 (1980).

[9] P.H. Frampton, Phys. Lett. **88B**, 299 (1979).

[10] P.H. Frampton, Phys. Rev. Lett. **69**, 2889 (1992); J. Agrawal, P.H. Frampton and D. Ng, Nucl. Phys. **B386**, 267 (1992).

[11] A $SU(5) \otimes SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ model has recently been proposed in a differ-

ent perspective; C.D. Froggatt, D.J. Smith and H.B. Nielsen, *GUTPA-95-01-1*, hep-ph/9603436. Our approach here is partially inspired by the model.

[12] O.C.W. Kong, *IFP-729-UNC*, hep-ph/9608246.

[13] This is basically the mass term used previously in a $SU(3)_H$ model; M. Soldate, M.H. Reno and C.T. Hill, Phys. Lett. **B179**, 95 (1986).

[14] B. McWilliams and L.-F. Li, Nucl. Phys. **B179**, 62 (1981); O. Shanker, Nucl. Phys. **B206**, 253 (1982).

[15] R. Cahn and H. Harari, Nucl. Phys. **B176**, 135 (1980); J.L. Ritchie and S.G. Wojcicki, Rev Mod. Phys. **65**, 1149 (1993).

[16] N. Polonsky, *LMU-TPW-96-16* and references therein.

[17] For the analysis of using an extra vector-like down-type quark to fix the R_b anomaly, see C.-H.V. Chang, D. Chang and W.-Y. Keung, NHCU-HEP-96-1, hep-ph/9601326;

Table Caption.

Table 1: The $SU(4)_A \otimes SU(3)_C \otimes SU(2)_L \otimes U(1)_X$ model – representation structures, anomaly cancellations and the SM embedding.
The hypercharge (differ by a factor of 6 from the standard normalization) is given by $Y = 3/2Z - 1/2X$ where Y, Z and X are charges of correspondent $U(1)$ groups, with $U(1)_Z \otimes SU(3)_H \subset SU(4)_A$. $Q, \bar{u}, \bar{d}$ denote the quark multiplets; L and E multiplets with the quantum numbers of leptonic doublets and singlets; $S, \bar{S}$, Q' and $\bar{Q}'$, the vector-like singlets and quark doublets; and N a right-handed neutrino state.

Table 2: Suggestive representation structures from the standard model to $SU(N)_A \otimes SU(3)_C \otimes SU(2)_L \otimes U(1)_X$, $N = 4, 5$ and 6, with three families.
Here we suppress the $U(1)_X$-charges.

Table 1: The $SU(4)_A \otimes SU(3)_C \otimes SU(2)_L \otimes U(1)_X$ model – representation structures, anomaly cancellations and the hypercharge embedding.

$SU(4)_A \otimes SU(3)_C \otimes SU(2)_L$ Rep.	$U(1)_X$	Gauge anomalies					$U(1)_Y$ states
		$U(1)$-grav.	$[SU(4)]^2U(1)$	$[SU(3)]^2U(1)$	$[SU(2)]^2U(1)$	$[U(1)]^3$	
$(\mathbf{4},\mathbf{3},\mathbf{2})$	**1**	24	6	8	12	24	3 $\mathbf{1}(Q)$ $\mathbf{-5}(Q^{'})$
$(\bar{\mathbf{4}},\bar{\mathbf{3}},\mathbf{1})$	**5**	60	15	20		1500	3 $\mathbf{-4}(\bar{u})$ $\mathbf{2}(\bar{d})$
$(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$	**3**	24	6		12	216	3 $\mathbf{-3}(L)$ $\mathbf{3}(\bar{L})$
$(\bar{\mathbf{4}},\mathbf{1},\mathbf{1})$	**9**	36	9			2916	3 $\mathbf{-6}(\bar{E})$ $\mathbf{0}(N)$
$(\mathbf{6},\mathbf{1},\mathbf{1})$	**-18**	-108	-36			-34992	3 $\mathbf{6}(E)$ 3 $\mathbf{12}(S)$
$(\mathbf{1},\bar{\mathbf{3}},\mathbf{2})$	**-10**	-60		-20	-30	-6000	$\mathbf{5}(\bar{Q}^{'})$
$(\mathbf{1},\bar{\mathbf{3}},\mathbf{1})$	**-4**	-12		-4		-192	$\mathbf{2}(\bar{d})$
$(\mathbf{1},\bar{\mathbf{3}},\mathbf{1})$	**-4**	-12		-4		-192	$\mathbf{2}(\bar{d})$
$(\mathbf{1},\mathbf{1},\mathbf{2})$	**6**	12			6	432	$\mathbf{-3}(L)$
3 $(\mathbf{1},\mathbf{1},\mathbf{1})$	**24**	72				41472	3 $\mathbf{-12}(\bar{S})$
3 $(\mathbf{1},\mathbf{1},\mathbf{1})$	**-12**	-36				-5184	3 $\mathbf{6}(E)$
	Total	0	0	0	0	0	

Table 2: Suggestive representation structures from the standard model to $SU(N)_A \otimes SU(3)_C \otimes SU(2)_L \otimes U(1)_X$, $N = 4, 5$ and 6, with three families.

SM	$N=4$	$N=5$	$N=6$
	$(\mathbf{4},\mathbf{3},\mathbf{2})$	$(\mathbf{5},\mathbf{3},\mathbf{2})$	$(\mathbf{6},\mathbf{3},\mathbf{2})$
	$(\bar{\mathbf{4}},\bar{\mathbf{3}},\mathbf{1})$	$(\bar{\mathbf{5}},\bar{\mathbf{3}},\mathbf{1})$	$(\bar{\mathbf{6}},\bar{\mathbf{3}},\mathbf{1})$
	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$	$(\bar{\mathbf{5}},\mathbf{1},\mathbf{2})$	$(\bar{\mathbf{6}},\mathbf{1},\mathbf{2})$
	$(\bar{\mathbf{4}},\mathbf{1},\mathbf{1})$	$(\bar{\mathbf{5}},\mathbf{1},\mathbf{1})$	$(\bar{\mathbf{6}},\mathbf{1},\mathbf{1})$
$(\mathbf{3},\mathbf{2})$	$(\mathbf{1},\bar{\mathbf{3}},\mathbf{2})$	$(\mathbf{1},\bar{\mathbf{3}},\mathbf{2})$	$(\mathbf{1},\bar{\mathbf{3}},\mathbf{2})$
		$(\mathbf{1},\bar{\mathbf{3}},\mathbf{2})$	$(\mathbf{1},\bar{\mathbf{3}},\mathbf{2})$
			$(\mathbf{1},\bar{\mathbf{3}},\mathbf{2})$
$(\bar{\mathbf{3}},\mathbf{1})$	$(\mathbf{1},\bar{\mathbf{3}},\mathbf{1})$	$(\mathbf{1},\bar{\mathbf{3}},\mathbf{1})$	
$(\bar{\mathbf{3}},\mathbf{1})$	$(\mathbf{1},\bar{\mathbf{3}},\mathbf{1})$		
$(\mathbf{1},\mathbf{2})$	$(\mathbf{1},\mathbf{1},\mathbf{2})$		$(\mathbf{1},\mathbf{1},\mathbf{2})$
$(\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
$\#Q = 1(\times 3)$	$\#Q = 4 - 1$	$\#Q = 5 - 2$	$\#Q = 6 - 3$
$\#\bar{q} = 2(\times 3)$	$\#\bar{q} = 4 + 2$	$\#\bar{q} = 5 + 1$	$\#\bar{q} = 6$
$\#L = 1(\times 3)$	$\#L = 4 \pm 1$	$\#L = 5$	$\#L = 6 \pm 1$